A Neural Network Approach for High-Dimensional Optimal Control

OT and MFG Seminar

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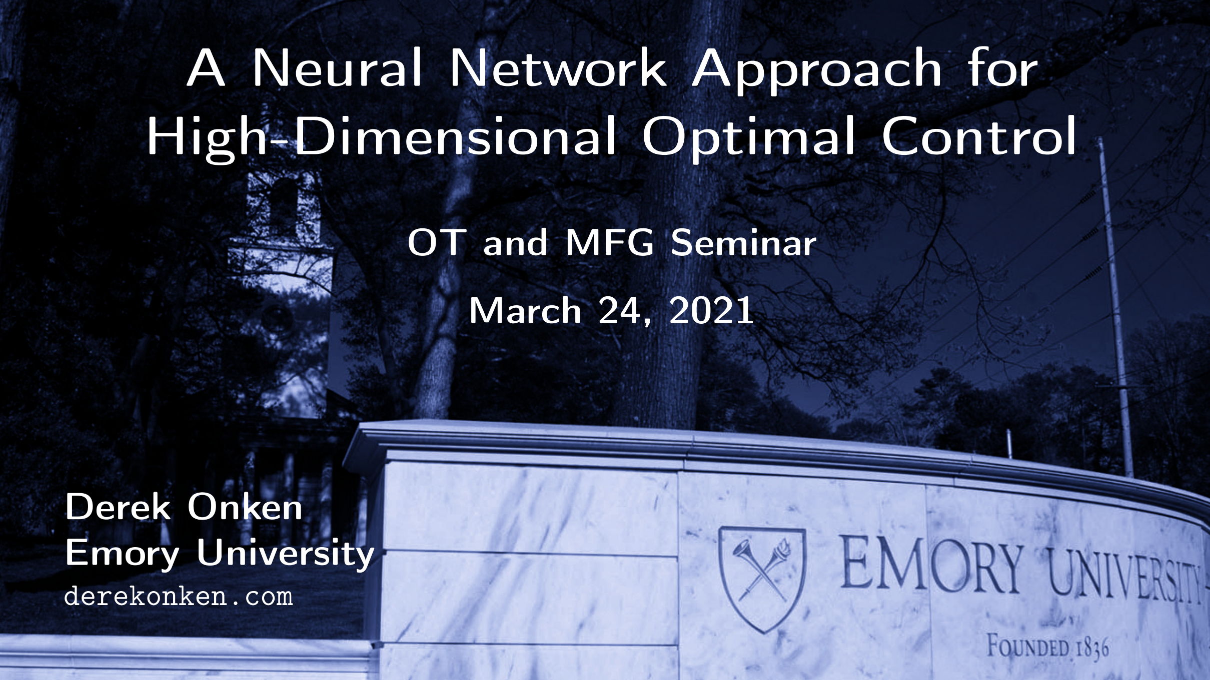
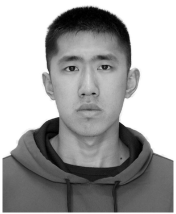


Collaborators and Acknowledgments



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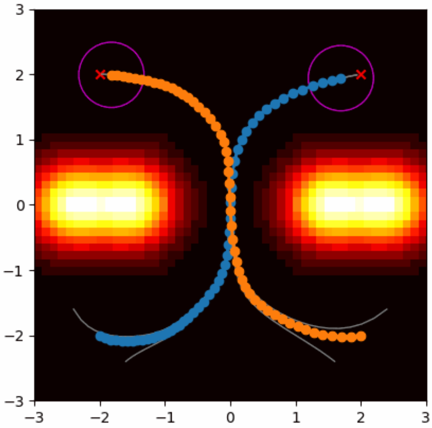
Overview

- **Background**
  - Problem
  - Pontryagin Maximum Principle (PMP)
  - Hamilton–Jacobi–Bellman Partial Differential Equation (HJB)
- **Mathematical Formulation**
  - Shock-Robustness
  - HJB Penalizers
- **Neural Networks (NNs)**
  - Model Formulation
  - Numerics
- **Results**
  - 150-Dimensional Swarm Trajectory Planning
  - Quadcopter with Complicated Dynamics
- **Conclusion**

Optimal Control (OC) Problem

Corridor Problem

Consider two *centrally-controlled* agents that navigate through a corridor/valley between two hills to fixed targets



Assume

- We have control over the agents' velocities (the *control*)

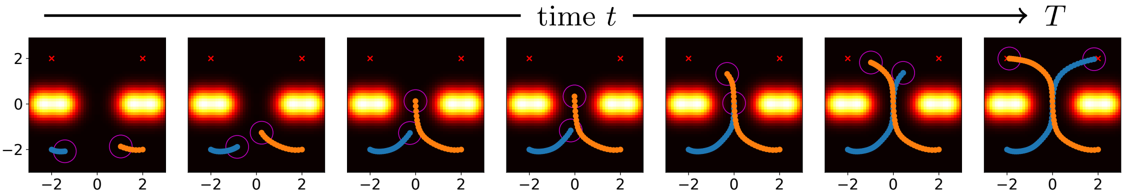
Want

- Shortest paths, e.g. the geodesics (*optimality*)
- No collisions
- Agents to reach targets at final time

Multi-Agent Formulation

Consider n agents initially at $x_1, \ldots, x_n \in \mathbb{R}^q \quad \Rightarrow \boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^d$

Agents follow trajectories $\boldsymbol{z}_{\boldsymbol{x}}(t)$ during time $t \in [0, T]$



Initial

$$\boldsymbol{z}_{\boldsymbol{x}}(0) = \boldsymbol{x} = \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} -2 \\ -2 \end{matrix}}\right\} \text{agent } 1 \\ \left.\vphantom{\begin{matrix} 2 \\ -2 \end{matrix}}\right\} \text{agent } 2 \end{matrix}$$

Target

$$\boldsymbol{y} = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 2 \end{bmatrix}$$

Terminal Cost

$$G\big(\boldsymbol{z}_{\boldsymbol{x}}(T)\big) = \frac{\alpha_1}{2} \|\boldsymbol{z}_{\boldsymbol{x}}(T) - \boldsymbol{y}\|^2$$

for multiplier $\alpha_1 \in \mathbb{R}$

Trajectories Governed by Differential Equation

The state $\boldsymbol{z_x}$ depends on the control $\boldsymbol{u_x}$ and previous state via the system

$$\partial_t \boldsymbol{z_x}(t) = f\big(t, \boldsymbol{z_x}(t), \boldsymbol{u_x}(t)\big), \quad \boldsymbol{z_x}(0) = \boldsymbol{x} \tag{1}$$

For Corridor: $= \boldsymbol{u_x}(t)$ (the velocity)

where

- time $t \in [0, T]$
- initial state $\boldsymbol{x} \in \mathbb{R}^d$
- admissible controls $U \subset \mathbb{R}^a$
- $f\colon [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$ models the evolution of the state $\boldsymbol{z_x}\colon [0,T] \to \mathbb{R}^d$ in response to the control $\boldsymbol{u_x}\colon [0,T] \to U$

Running Cost

Running costs where z_i and u_i are the state and control for the ith agent, respectively

$$L\big(t, \boldsymbol{z}(t), \boldsymbol{u}(t)\big) = E\big(\boldsymbol{z}(t), \boldsymbol{u}(t)\big) + \alpha_2 Q\big(\boldsymbol{z}(t), \boldsymbol{u}(t)\big) + \alpha_3 W\big(\boldsymbol{z}(t), \boldsymbol{u}(t)\big)$$

$$= \sum_{i=1}^{n} \underbrace{E_i\big(z_i(t), u_i(t)\big)}_{\text{For Corridor: } \frac{1}{2}\|u_i(t)\|^2} + \alpha_2 \sum_{i=1}^{n} \underbrace{Q_i\big(z_i(t), u_i(t)\big)}_{\text{sum of Gaussians}} + \alpha_3 \sum_{j \neq i} \underbrace{W_{ij}\big(z_i(t), z_j(t)\big)}_{\text{piecewise Gaussian repulsion}}$$

for multipliers $\alpha_2, \alpha_3 \in \mathbb{R}$ and

- E_i is the energy of an agent,
- Q_i represents any obstacles or terrain,
- W_{ij} are the interaction costs between homogeneous agents i and j with radius r

$$W_{ij}(z_i, z_j) = \begin{cases} \exp\left(-\frac{\|z_i - z_j\|_2^2}{2r^2}\right), & \|z_i - z_j\|_2 < 2r \\ 0, & \text{otherwise} \end{cases}$$

Optimal Control (OC) Problem

Running Cost: $L(s,\cdot) = E(\cdot) + \alpha_2 Q(\cdot) + \alpha_3 W(\cdot)$
Terminal Cost: $G\big(\boldsymbol{z_x}(T)\big) = \frac{\alpha_1}{2}\|\boldsymbol{z_x}(T) - \boldsymbol{y}\|^2$

Goal: Find the control that incurs minimal cost[1]

$$\Phi(t,\boldsymbol{x}) = \inf_{\boldsymbol{u_x}} \left\{ \int_t^T L\big(s, \boldsymbol{z_x}(s), \boldsymbol{u_x}(s)\big)\, \mathrm{d}s + G\big(\boldsymbol{z_x}(T)\big) \right\} \tag{2}$$

- $\Phi(t,\boldsymbol{x}) \in \mathbb{R}$ is the *value function* (i.e., optimal cost-to-go)
- solution $\boldsymbol{u}_{\boldsymbol{x}}^*$ is the *optimal control*
- *optimal trajectory* $\boldsymbol{z}_{\boldsymbol{x}}^*$ dictated by $\boldsymbol{u}_{\boldsymbol{x}}^*$

[1] Fleming and Soner. *Controlled Markov Processes and Viscosity Solutions*. 2006.

Pontryagin Maximum Principle (PMP)

Existing Approach

Solve the forward-backward system[2] for $0 \le t \le T$

$$\begin{cases} \partial_t \boldsymbol{z}_{\boldsymbol{x}}^*(t) = -\nabla_{\boldsymbol{p}} H\big(t, \boldsymbol{z}_{\boldsymbol{x}}^*(t), \boldsymbol{p}_{\boldsymbol{x}}(t)\big), \\ \partial_t \boldsymbol{p}_{\boldsymbol{x}}(t) = \nabla_{\boldsymbol{x}} H\big(t, \boldsymbol{z}_{\boldsymbol{x}}^*(t), \boldsymbol{p}_{\boldsymbol{x}}(t)\big), \\ \boldsymbol{z}_{\boldsymbol{x}}^*(0) = \boldsymbol{x}, \quad \boldsymbol{p}_{\boldsymbol{x}}(T) = \nabla G\big(\boldsymbol{z}_{\boldsymbol{x}}^*(T)\big), \end{cases} \tag{3}$$

where

- Hamiltonian $H(t, \boldsymbol{x}, \boldsymbol{p}_{\boldsymbol{x}}) = \sup_{\boldsymbol{u}_{\boldsymbol{x}} \in U} \{-\boldsymbol{p}_{\boldsymbol{x}} \cdot f(t, \boldsymbol{x}, \boldsymbol{u}_{\boldsymbol{x}}) - L(t, \boldsymbol{x}, \boldsymbol{u}_{\boldsymbol{x}})\}$
- adjoint $\boldsymbol{p}_{\boldsymbol{x}} \colon [0, T] \to \mathbb{R}^d$

then notation-wise, we have $\boldsymbol{u}_{\boldsymbol{x}}^*(t) = \boldsymbol{u}^*\big(t, \boldsymbol{z}_{\boldsymbol{x}}^*(t), \boldsymbol{p}_{\boldsymbol{x}}(t)\big)$

[2] Pontryagin et al. *The Mathematical Theory of Optimal Processes*. 1962.

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where

- Hamiltonian $H(t, \boldsymbol{x}, \boldsymbol{p}_{\boldsymbol{x}}) = \sup_{\boldsymbol{u}_{\boldsymbol{x}} \in U} \{-\boldsymbol{p}_{\boldsymbol{x}} \cdot f(t, \boldsymbol{x}, \boldsymbol{u}_{\boldsymbol{x}}) - L(t, \boldsymbol{x}, \boldsymbol{u}_{\boldsymbol{x}})\}$
- adjoint $\boldsymbol{p}_{\boldsymbol{x}} : [0, T] \to \mathbb{R}^d$

then notation-wise, we have $\boldsymbol{u}_{\boldsymbol{x}}^*(t) = \boldsymbol{u}^*\big(t, \boldsymbol{z}_{\boldsymbol{x}}^*(t), \boldsymbol{p}_{\boldsymbol{x}}(t)\big)$

Comments

- *Local* solution method
  - Solved for a single $\boldsymbol{x}$
  - For a new $\boldsymbol{x}$, need to resolve (3)
- Solving the system is difficult and depends on the initial guess $\boldsymbol{p}_{\boldsymbol{x}}(0)$ (if using a shooting method)

[2]Pontryagin et al. *The Mathematical Theory of Optimal Processes*. 1962.

Hamilton-Jacobi-Bellman (HJB)

Existing Approach

Solve the HJB PDE[3]
(also called *dynamic programming* equations)

$$\begin{cases} -\partial_t \Phi(t, \boldsymbol{x}) = -H\big(t, \boldsymbol{x}, \nabla\Phi(t, \boldsymbol{x})\big), \\ \Phi(T, \boldsymbol{x}) = G(\boldsymbol{x}) \end{cases} \tag{4}$$

arises from correspondence

$$\boldsymbol{p}_{\boldsymbol{x}}(t) = \nabla\Phi\big(t, \boldsymbol{z}^*_{\boldsymbol{x}}(t)\big) \tag{5}$$

[3] Bellman. *Dynamic Programming*. 1957.

Hamilton-Jacobi-Bellman (HJB)

Existing Approach

Solve the HJB PDE[3]
(also called *dynamic programming* equations)

$$\begin{cases} -\partial_t \Phi(t, \boldsymbol{x}) = -H\big(t, \boldsymbol{x}, \nabla \Phi(t, \boldsymbol{x})\big), \\ \Phi(T, \boldsymbol{x}) = G(\boldsymbol{x}) \end{cases} \tag{4}$$

arises from correspondence

$$\boldsymbol{p}_{\boldsymbol{x}}(t) = \nabla \Phi\big(t, \boldsymbol{z}^*_{\boldsymbol{x}}(t)\big) \tag{5}$$

Comments

- *Global* solution method
  - Solved for all $\boldsymbol{x}$
  - For a new $\boldsymbol{x}$, no recomputation
- Need grids to solve (4), which scale poorly to high-dimensions

[3] Bellman. *Dynamic Programming*. 1957.

Our Approach

Motivation

Want:

- Semi-global solution method (from HJB)

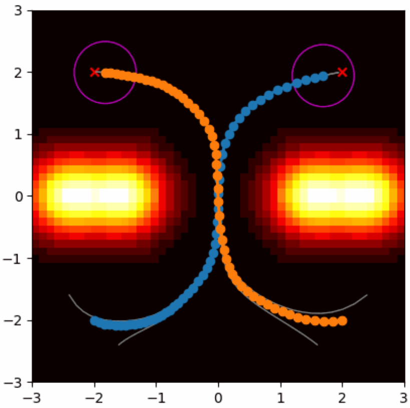
  $\Rightarrow$ one model useful for many initial conditions

  $\Rightarrow$ method is robust to shocks/disturbances

- High-dimensional (from PMP)

  $\Rightarrow$ multi-agent problems provide high dimensionality and are easy to visualize

Corridor Problem



Semi-Global Solution Method

Robust to Shocks

Want: semi-global Φ (value function)

How to obtain:

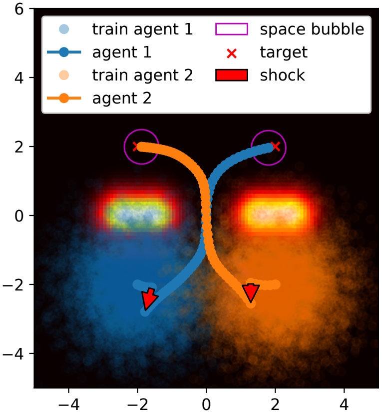
- Solve for Hamiltonian H
- Replace adjoint $\boldsymbol{p}$ with $\nabla\Phi$ using (5)
- Use initial states sampled from Gaussian distribution
- Solve

$$\min_{\Phi} \mathbb{E}_{\boldsymbol{x}\sim\mathcal{N}(\mu,\boldsymbol{\Sigma})} \left\{ \int_0^T L\big(s, \boldsymbol{z_x}(s), \boldsymbol{u_x}(s)\big)\, \mathrm{d}s + G\big(\boldsymbol{z_x}(T)\big) \right\}$$

s.t.

$$\partial_t \boldsymbol{z_x}(t) = -\nabla_{\boldsymbol{p}} H\big(t, \boldsymbol{z_x}(t), \nabla\Phi(t, \boldsymbol{z_x}(t))\big) = -\nabla\Phi(t, \boldsymbol{z_x}(t))$$

For Corridor



Example:

$$\mu = \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \boldsymbol{I}$$

Penalizers

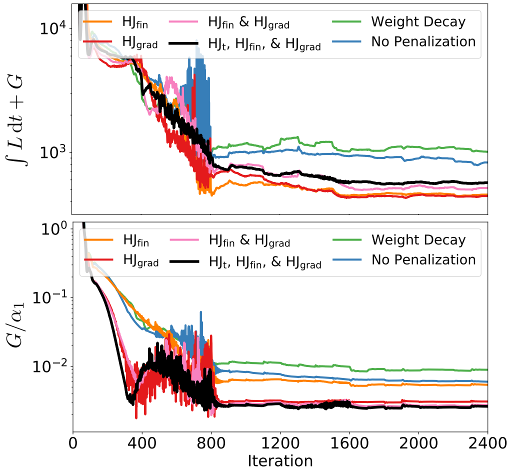
Recall the HJB equations

$$-\partial_t \Phi\big(t, \boldsymbol{z_x}(t)\big) = -H\big(t, \boldsymbol{z_x}(t), \nabla\Phi(t, \boldsymbol{z_x}(t))\big),$$
$$\Phi\big(T, \boldsymbol{z_x}(T)\big) = G\big(\boldsymbol{z_x}(T)\big)$$

Make penalizers

$$c_{\mathrm{HJt},\boldsymbol{x}}(t) =$$
$$\int_0^t \Big| \partial_s \Phi(s, \boldsymbol{z_x}(s)) - H\big(s, \boldsymbol{z_x}(s), \nabla\Phi(s, \boldsymbol{z_x}(s))\big) \Big| \, \mathrm{d}s$$
$$c_{\mathrm{HJfin},\boldsymbol{x}} = \Big| \Phi(T, \boldsymbol{z_x}(T)) \; - \; G(\boldsymbol{z_x}(T)) \Big|$$
$$c_{\mathrm{HJgrad},\boldsymbol{x}} = \Big| \nabla\Phi(T, \boldsymbol{z_x}(T)) \; - \; \nabla G(\boldsymbol{z_x}(T)) \Big|$$

Empirically Effective in Training



HJt penalizer ⇒ few time steps[4,5]

[4] Yang and Karniadakis. "Potential Flow Generator with L_2 Optimal Transport . . . ". 2020.
[5] Onken et al. "OT-Flow: Fast and Accurate Continuous Normalizing Flows via Optimal Transport". 2020.

Formulation

Rewrite time-integrals as part of the ODE

$$\min_{\Phi} \mathop{\mathbb{E}}_{\boldsymbol{x}\sim\mathcal{N}(\mu,\boldsymbol{\Sigma})} c_{\mathrm{L},\boldsymbol{x}}(T) + G(\boldsymbol{z}_{\boldsymbol{x}}(T)) + \beta_1 c_{\mathrm{HJt},\boldsymbol{x}}(T) + \beta_2\, c_{\mathrm{HJfin},\boldsymbol{x}} + \beta_3\, c_{\mathrm{HJgrad},\boldsymbol{x}}, \tag{6}$$

subject to

$$\partial_t \begin{pmatrix} \boldsymbol{z}_{\boldsymbol{x}}(t) \\ c_{\mathrm{L},\boldsymbol{x}}(t) \\ c_{\mathrm{HJt},\boldsymbol{x}}(t) \end{pmatrix} = \begin{pmatrix} -\nabla_{\boldsymbol{p}} H\big(t, \boldsymbol{z}_{\boldsymbol{x}}(t), \nabla\Phi(t, \boldsymbol{z}_{\boldsymbol{x}}(t))\big) \\ L_{\boldsymbol{x}}(t) \\ \Big| \partial_t \Phi(t, \boldsymbol{z}_{\boldsymbol{x}}(t)) - H\big(t, \boldsymbol{z}_{\boldsymbol{x}}(t), \nabla\Phi(t, \boldsymbol{z}_{\boldsymbol{x}}(t))\big) \Big| \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{z}_{\boldsymbol{x}}(0) \\ c_{\mathrm{L},\boldsymbol{x}}(0) \\ c_{\mathrm{HJt},\boldsymbol{x}}(0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{x} \\ 0 \\ 0 \end{pmatrix}.$$

where, by the envelope formula,

$$L_{\boldsymbol{x}}(t) = \nabla\Phi(t, \boldsymbol{z}_{\boldsymbol{x}}(t)) \cdot \nabla_{\boldsymbol{p}} H\big(t, \boldsymbol{z}_{\boldsymbol{x}}(t), \nabla\Phi(t, \boldsymbol{z}_{\boldsymbol{x}}(t))\big) - H\big(t, \boldsymbol{z}_{\boldsymbol{x}}(t), \nabla\Phi(t, \boldsymbol{z}_{\boldsymbol{x}}(t))\big)$$

Scalars $\beta_1, \beta_2, \beta_3$ are weighted multipliers (NN hyperparameters)

How do we solve this PDE-constrained optimization problem?

How do we solve this PDE-constrained optimization problem?

Blend Neural Networks and Differential Equations

Choose your buzzword: Neural ODEs, Physics-Informed Neural Networks, etc.

Neural Network (NN) Basics

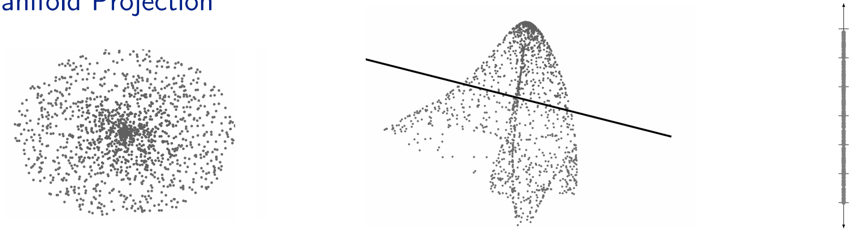
Consider a parameterized function:

$$C = g(\boldsymbol{z}; \boldsymbol{\theta})$$

where

- $\boldsymbol{z} \in \mathbb{R}^d$ is an input item (e.g., the state of the system)
- $C \in \mathbb{R}$ is the corresponding output (e.g., the value from Φ)
- $\boldsymbol{\theta} \in \mathbb{R}^p$ are the parameters/weights of the model g

Think: Manifold Projection



Input Features — Transformed (Hidden) Features — Output

Single-Layer Example

d - # features

m - width

Features

$\boldsymbol{z} \in \mathbb{R}^d$

Weights ($\boldsymbol{\theta}$)

$\boldsymbol{K} \in \mathbb{R}^{m \times d}$

$\boldsymbol{w} \in \mathbb{R}^m$

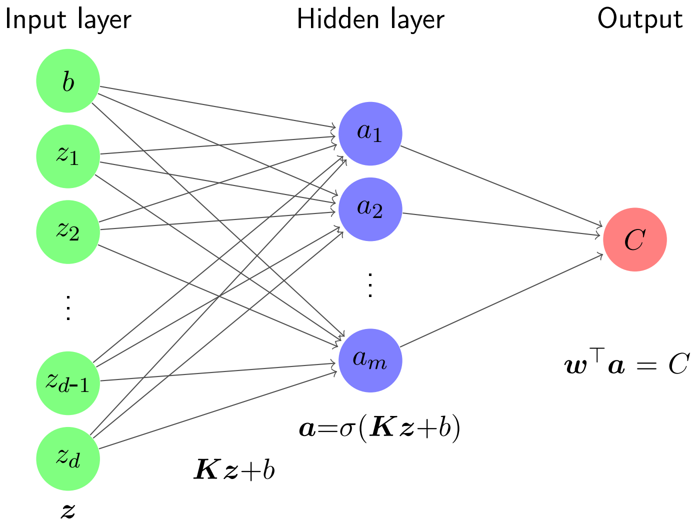
bias $b \in \mathbb{R}$

Outputs

$C \in \mathbb{R}$

Nonlinearity σ

$\tanh$, sigmoid, etc.



Our Network

A Brief Look Under the Hood

We parameterize the value function

$$\boldsymbol{a}_0 = \sigma(\boldsymbol{K}_0 \boldsymbol{s} + \boldsymbol{b}_0),$$

- space-time inputs $\boldsymbol{s} = (\boldsymbol{x}, t) \in \mathbb{R}^{d+1}$

[6] He et al. "Deep Residual Learning for Image Recognition". 2016.

Our Network

A Brief Look Under the Hood

We parameterize the value function

where $N(\boldsymbol{s}) = \boldsymbol{a}_0 + \sigma(\boldsymbol{K}_1 \boldsymbol{a}_0 + \boldsymbol{b}_1)$,

$$\boldsymbol{a}_0 = \sigma(\boldsymbol{K}_0 \boldsymbol{s} + \boldsymbol{b}_0),$$

and

- space-time inputs $\boldsymbol{s}=(\boldsymbol{x}, t) \in \mathbb{R}^{d+1}$
- $N(\boldsymbol{s})\colon \mathbb{R}^{d+1} \to \mathbb{R}^m$ is a residual neural network (ResNet)[6]
- element-wise activation function $\sigma(\boldsymbol{x}) = \log(\exp(\boldsymbol{x}) + \exp(-\boldsymbol{x}))$

[6] He et al. "Deep Residual Learning for Image Recognition". 2016.

Our Network

A Brief Look Under the Hood

We parameterize the value function with

$$\Phi(\boldsymbol{s};\boldsymbol{\theta}) = \boldsymbol{w}^\top N(\boldsymbol{s}) + \frac{1}{2}\boldsymbol{s}^\top(\boldsymbol{A}^\top\boldsymbol{A})\boldsymbol{s} + \boldsymbol{b}^\top\boldsymbol{s} + c, \qquad \text{for} \quad \boldsymbol{\theta} = (\boldsymbol{w}, \boldsymbol{A}, \boldsymbol{b}, c, \boldsymbol{K}_0, \boldsymbol{K}_1, \boldsymbol{b}_0, \boldsymbol{b}_1)$$

where $N(\boldsymbol{s}) = \boldsymbol{a}_0 + \sigma(\boldsymbol{K}_1\boldsymbol{a}_0 + \boldsymbol{b}_1)$,

$$\boldsymbol{a}_0 = \sigma(\boldsymbol{K}_0\boldsymbol{s} + \boldsymbol{b}_0),$$

and

- space-time inputs $\boldsymbol{s}{=}(\boldsymbol{x}, t) \in \mathbb{R}^{d+1}$
- $N(\boldsymbol{s})\colon \mathbb{R}^{d+1} \to \mathbb{R}^m$ is a residual neural network (ResNet)[6]
- element-wise activation function $\sigma(\boldsymbol{x}) = \log(\exp(\boldsymbol{x}) + \exp(-\boldsymbol{x}))$
- $\boldsymbol{\theta}$ contains the trainable weights: $\boldsymbol{w} \in \mathbb{R}^m$, $\boldsymbol{A} \in \mathbb{R}^{10\times(d+1)}$, $\boldsymbol{b} \in \mathbb{R}^{d+1}$, $c \in \mathbb{R}$, $\boldsymbol{K}_0 \in \mathbb{R}^{m\times(d+1)}$, $\boldsymbol{K}_1 \in \mathbb{R}^{m\times m}$, and $\boldsymbol{b}_0, \boldsymbol{b}_1 \in \mathbb{R}^m$.

[6] He et al. "Deep Residual Learning for Image Recognition". 2016.

Differential Equations

Recall: We are solving

$$\min_{\Phi} \mathop{\mathbb{E}}_{\boldsymbol{x}\sim\mathcal{N}(\mu,\boldsymbol{\Sigma})} c_{\mathrm{L},\boldsymbol{x}}(T) + G(\boldsymbol{z}_{\boldsymbol{x}}(T)) + \beta_1 c_{\mathrm{HJt},\boldsymbol{x}}(T) + \beta_2\, c_{\mathrm{HJfin},\boldsymbol{x}} + \beta_3\, c_{\mathrm{HJgrad},\boldsymbol{x}},$$

subject to

$$\partial_t \begin{pmatrix} \boldsymbol{z}_{\boldsymbol{x}}(t) \\ c_{\mathrm{L},\boldsymbol{x}}(t) \\ c_{\mathrm{HJt},\boldsymbol{x}}(t) \end{pmatrix} = \begin{pmatrix} -\nabla_{\boldsymbol{p}} H\big(t, \boldsymbol{z}_{\boldsymbol{x}}(t), \nabla\Phi(t, \boldsymbol{z}_{\boldsymbol{x}}(t))\big) \\ L_{\boldsymbol{x}}(t) \\ \Big| \partial_t \Phi(t, \boldsymbol{z}_{\boldsymbol{x}}(t)) - H\big(t, \boldsymbol{z}_{\boldsymbol{x}}(t), \nabla\Phi(t, \boldsymbol{z}_{\boldsymbol{x}}(t))\big) \Big| \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{z}_{\boldsymbol{x}}(0) \\ c_{\mathrm{L},\boldsymbol{x}}(0) \\ c_{\mathrm{HJt},\boldsymbol{x}}(0) \end{pmatrix}, = \begin{pmatrix} \boldsymbol{x} \\ 0 \\ 0 \end{pmatrix}.$$

Differential Equations

Which is the same as training the neural ODE

$$\min_{\boldsymbol{\theta}} \mathop{\mathbb{E}}_{\boldsymbol{x}\sim\mathcal{N}(\mu,\boldsymbol{\Sigma})} c_{\mathrm{L},\boldsymbol{x}}(T) + G(\boldsymbol{z}_{\boldsymbol{x}}(T)) + \beta_1 c_{\mathrm{HJt},\boldsymbol{x}}(T) + \beta_2\, c_{\mathrm{HJfin},\boldsymbol{x}} + \beta_3\, c_{\mathrm{HJgrad},\boldsymbol{x}},$$

subject to

$$\partial_t \begin{pmatrix} \boldsymbol{z}_{\boldsymbol{x}}(t) \\ c_{\mathrm{L},\boldsymbol{x}}(t) \\ c_{\mathrm{HJt},\boldsymbol{x}}(t) \end{pmatrix} = F\big(t,\, \boldsymbol{z}_{\boldsymbol{x}}(t),\, \nabla\Phi(t, \boldsymbol{z}_{\boldsymbol{x}}(t)\,;\boldsymbol{\theta})\big), \quad \begin{pmatrix} \boldsymbol{z}_{\boldsymbol{x}}(0) \\ c_{\mathrm{L},\boldsymbol{x}}(0) \\ c_{\mathrm{HJt},\boldsymbol{x}}(0) \end{pmatrix}, = \begin{pmatrix} \boldsymbol{x} \\ 0 \\ 0 \end{pmatrix}.$$

Training and Numerics

Solving the Minimiziation / Training the Neural ODE:

Iterate through

1. Solve the ODE
2. Compute the loss function
3. Backpropagate
4. Update parameters $\boldsymbol{\theta}$

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ODE solver:

Runge-Kutta 4 $\Rightarrow$ efficient and accurate

Discretize-then-Optimize Approach:[7,8]

First, discretize the ODE at time points, then optimize over that discretization

As opposed to optimize-then-discretize, e.g., solve Karush-Kuhn-Tucker then discretize

[7] Gholaminejad, Keutzer, and Biros. "ANODE: Unconditionally Accurate Memory-Efficient . . . ". 2019.

[8] Onken and Ruthotto. "Discretize-Optimize vs. Optimize-Discretize for Time-Series . . . ". 2020.

Training and Numerics

Solving the Minimiziation / Training the Neural ODE:

Iterate through

1. Solve the ODE
2. Compute the loss function
3. Backpropagate
4. Update parameters $\boldsymbol{\theta}$

Loss / Objective Function:

$$J(\boldsymbol{\theta}) = \mathop{\mathbb{E}}_{\boldsymbol{x} \sim \mathcal{N}(\mu, \boldsymbol{\Sigma})} c_{\mathrm{L},\boldsymbol{x}}(T) + G(\boldsymbol{z}_{\boldsymbol{x}}(T)) + \beta_1 c_{\mathrm{HJt},\boldsymbol{x}}(T) + \beta_2\, c_{\mathrm{HJfin},\boldsymbol{x}} + \beta_3\, c_{\mathrm{HJgrad},\boldsymbol{x}}$$

Training and Numerics

Solving the Minimiziation / Training the Neural ODE:

Iterate through

1. Solve the ODE
2. Compute the loss function
3. Backpropagate
4. Update parameters $\boldsymbol{\theta}$

Compute gradient with respect to parameters (chain rule)

Use automatic differentiation[9] to compute $\nabla_{\boldsymbol{\theta}} J$

[9] Nocedal and Wright. *Numerical Optimization*. 2006.

Training and Numerics

Solving the Minimiziation / Training the Neural ODE:

Iterate through

1. Solve the ODE
2. Compute the loss function
3. Backpropagate
4. Update parameters $\boldsymbol{\theta}$

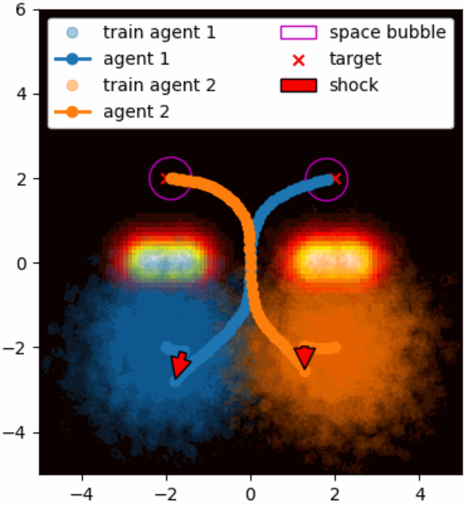
Use ADAM[10]

A stochastic subgradient method with momentum

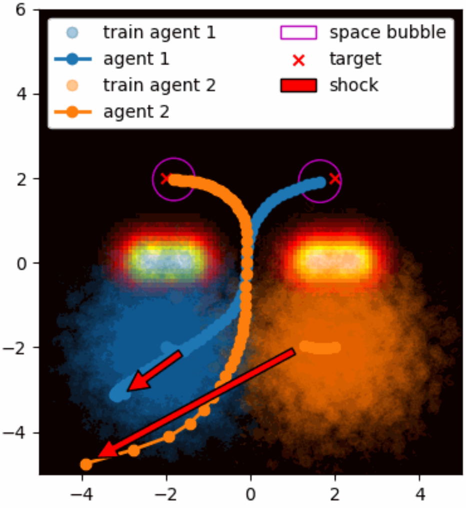
Empirically, ADAM works well in noisy high-dimensional spaces

[10] Kingma and Ba. "Adam: A Method for Stochastic Optimization". 2015.

Results



Small Shock



Large Shock

Baseline

Corridor

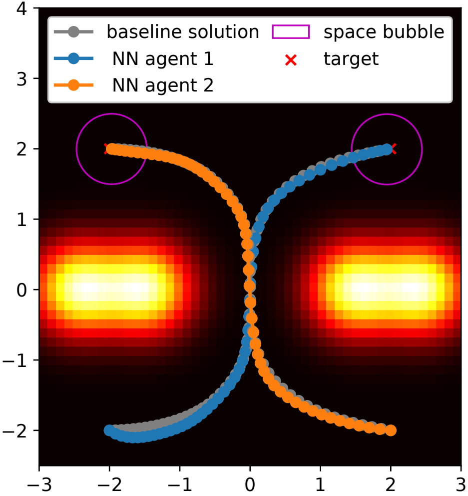
Running Cost: $L(t,\cdot) = E(\cdot) + \alpha_2 Q(\cdot) + \alpha_3 W(\cdot)$
Terminal Cost: $G(\boldsymbol{z}) = \frac{\alpha_1}{2}\|\boldsymbol{z}-\boldsymbol{y}\|^2$

Discrete optimization approach via forward Euler

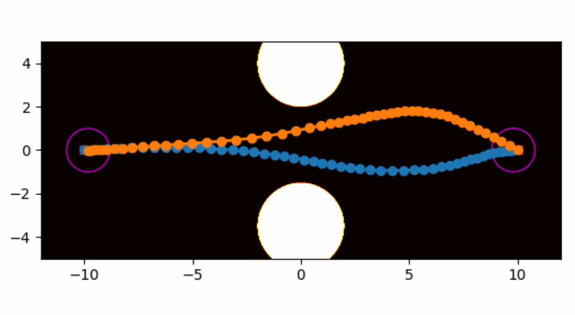
$$\min_{\{\boldsymbol{u}^{(k)}\}} \quad G\left(\boldsymbol{z}^{(n_t)}\right) + h\sum_{k=0}^{n_t-1} L\left(t^{(k)}, \boldsymbol{z}^{(k)}, \boldsymbol{u}^{(k)}\right)$$
$$\text{s.t.} \quad \boldsymbol{z}^{(k+1)} = \boldsymbol{z}^{(k)} + h\, f\left(t^{(k)}, \boldsymbol{z}^{(k)}, \boldsymbol{u}^{(k)}\right),$$
$$\boldsymbol{z}^{(0)} = \boldsymbol{x}$$

where $h{=}T/n_t$. We use $T{=}1$ and $n_t{=}50$.

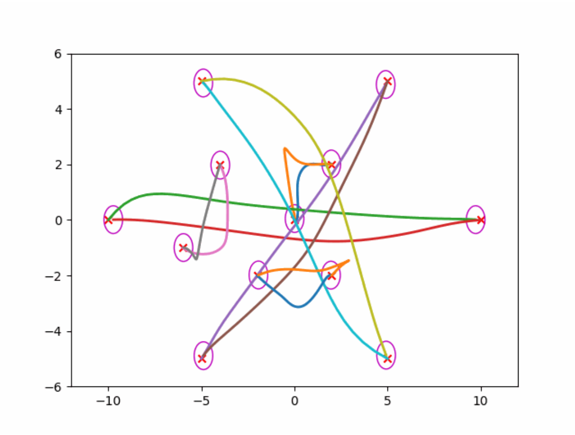
This is a *local* approach, whereas the NN is *global*



Swap Experiments



Two agents swap positions with hard corridor[11]



Twelve agents swap positions[11]

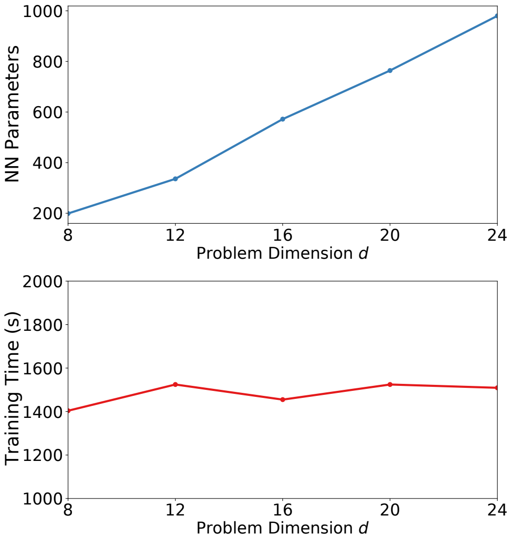
[11] Mylvaganam, Sassano, and Astolfi. "A Differential Game Approach to Multi-Agent Collision Avoidance". 2017.

Addressing Curse of Dimensionality[12]

Setup:

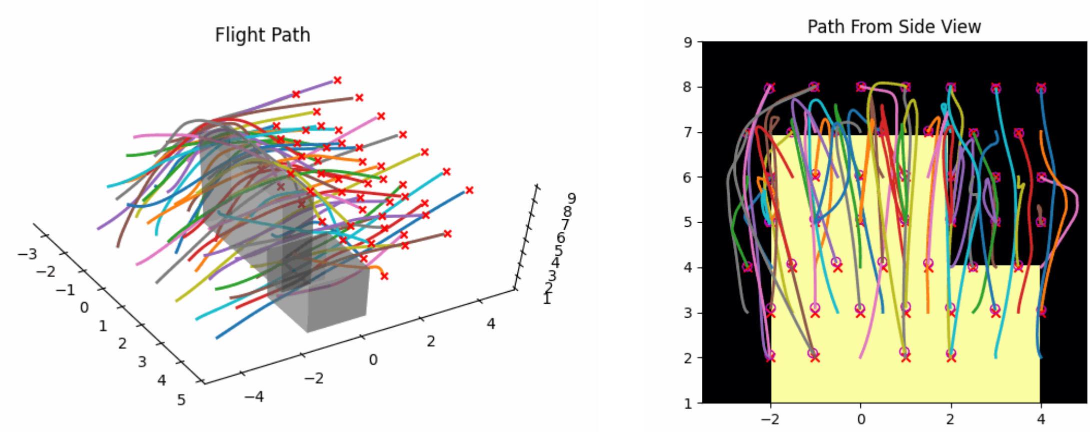
- Take subproblems of the 12-agent swap experiment (2, 3, 4, 5, and 6 pairs of agents)
- Train the smallest NN we can that achieves a fixed suboptimality (relative to baseline)

The number of parameters grows linearly with problem dimension d



[12] Bellman. *Dynamic Programming*. 1957.

Swarm Trajectory Planning



50 3-dimensional agents with obstacles[13]

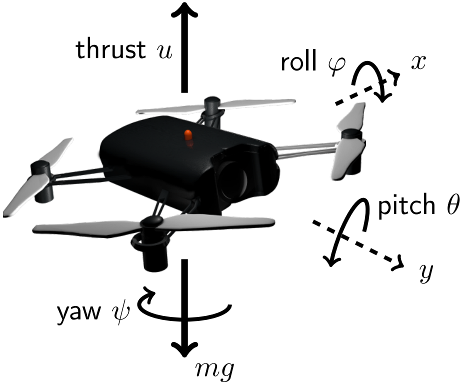
[13] Hönig et al. "Trajectory Planning for Quadrotor Swarms". 2018.

Quadcopter Problem

More complicated dynamics[14]

Controls: thrust u, torques $\tau_\psi, \tau_\theta, \tau_\varphi$

$$\dot{\boldsymbol{z}} = f(\boldsymbol{x}, \boldsymbol{u}) \implies \begin{cases} \dot{x} = v_x \\ \dot{y} = v_y \\ \dot{z} = v_z \\ \dot{\psi} = v_\psi \\ \dot{\theta} = v_\theta \\ \dot{\varphi} = v_\varphi \\ \dot{v}_x = \frac{u}{m} f_7(\psi, \theta, \varphi) \\ \dot{v}_y = \frac{u}{m} f_8(\psi, \theta, \varphi) \\ \dot{v}_z = \frac{u}{m} f_9(\theta, \varphi) - g \\ \dot{v}_\psi = \tau_\psi \\ \dot{v}_\theta = \tau_\theta \\ \dot{v}_\varphi = \tau_\varphi \end{cases}$$

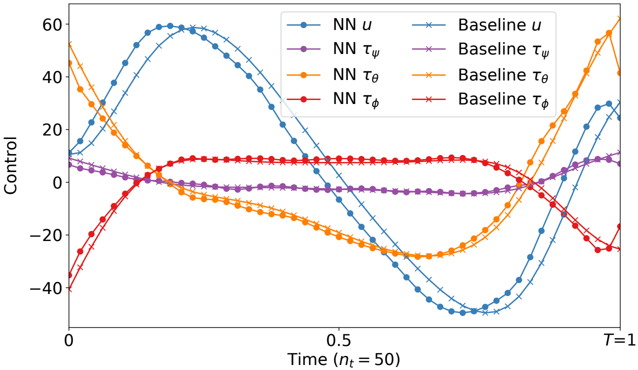


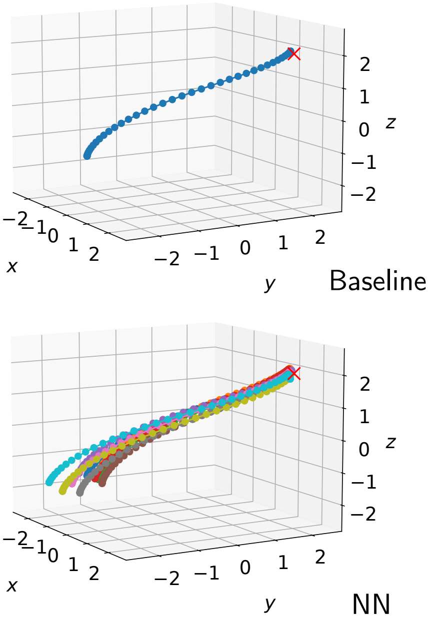
where

$$\begin{cases} f_7(\psi, \theta, \varphi) & = \sin(\psi)\sin(\varphi) + \cos(\psi)\sin(\theta)\cos(\varphi), \\ f_8(\psi, \theta, \varphi) & = -\cos(\psi)\sin(\varphi) + \sin(\psi)\sin(\theta)\cos(\varphi), \\ f_9(\theta, \varphi) & = \cos(\theta)\cos(\varphi). \end{cases}$$

[14] Carrillo et al. "Modeling the Quad-Rotor Mini-Rotorcraft". 2013.

Quadcopter Comparison with Baseline





Review

- **Want to solve**
  - High-Dimensional Control Problems
  - Semi-Globally
- **Combine Pontryagin Maximum Principle and Hamilton-Jacobi-Bellman approaches**
- **Parameterize the value function Φ with a neural network**
- **Solve trajectory problem in 150 dimensions**
- **Solve quadcopter problem with complicated dynamics**
- **Demonstrate shock-robustness**

Conclusions

- **Parameterizing Φ**
  $\Rightarrow$ extrapolation capabilities
- **HJB penalizers improve training**
- **Lagrangian coordinates (no grids) help scalability**

DO, L Nurbekyan, X Li, S Wu Fung, S Osher, L Ruthotto
A Neural Network Approach Applied to Multi-Agent Optimal Control
2021 European Control Conference
arXiv:2011.04757, 2020

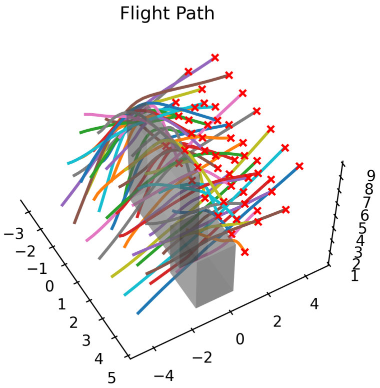
Coming Soon:

DO, L Nurbekyan, X Li, S Wu Fung, S Osher, L Ruthotto
A Neural Network Approach for High-Dimensional Optimal Control

Code: `github.com/donken/NeuralOC`
Simulations: `imgur.com/a/eWr6sUb`

Future Work

- **More rigorous experiments with many 12-d quadcopters**
- **Deployment on actual quadcopters**
- **Combination with existing methods and sensors**



Future Work

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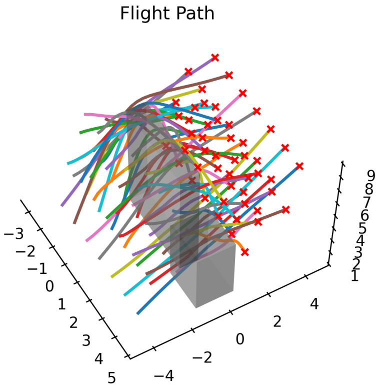

Questions?

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